Active Suppression of Traveling Waves in Structures

Jeffrey K. Bennighof* and Leonard Meirovitch†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia

This paper is concerned with the control of traveling waves in structures. Two approaches are used, independent modal-space control (IMSC) and direct feedback control. It is demonstrated that direct feedback control is more suitable for problems in which a large number of higher modes require control and IMSC is more attractive when the number of modes in need of control is not large or when inherent damping provides passive control, suppressing the higher modes. One fact worthy of notice is that although IMSC represents a global approach compared to direct feedback control, the actuator forces still tend to concentrate near localized disturbances such as traveling waves.

Introduction

TRAVELING waves are a type of disturbance in flexible structures in which a localized pulse or "ripple" passes through a structure.[1] There is a certain equivalence between motion in the form of traveling waves and vibration in the natural modes of a structure. On the one hand, any disturbance in a structure, including traveling waves, can be described in terms of the natural modes, if enough modes are retained in the description. This is so because the natural modes form a complete set, capable of describing any motion of the structure to any degree of accuracy. On the other hand, simple harmonic vibration in one of the natural modes can be regarded as a standing wave and can be described in terms of a pair of wave profiles traveling in opposite directions simultaneously.

When the interest lies in suppressing wave motion in a flexible structure, the question arises as to what type of control is likely to be more effective. In this paper, we investigate the performance of two techniques, modal control[2] and direct feedback control.[3,4] We also note that other techniques[5] have been proposed for traveling-wave suppression. Reference 5 presents several control concepts based on disturbance propagation theory.

In modal control, such as in the independent modal-space control (IMSC) method, a structure is controlled by controlling its modes. If we assume that the modes of a structure are known, the control task can be carried out by estimating modal displacements and velocities from sensor measurements, computing modal feedback control forces from these estimates, and obtaining actuator control forces from the modal forces by means of a linear transformation. Practical limitations on the number of sensors and actuators restrict the number of modal states that can be observed accurately and controlled independently.

The modes of a structure are global functions, defined over the entire domain of the structure. Hence, the question can be raised as to the suitability of representing localized disturbances, such as traveling waves, in terms of a limited number of global functions, and controlling such disturbances by means of a finite number of modal controls that are global in nature. One can envision a modal control system that senses a traveling wave and responds with actuator forces in areas of the structure not yet reached by the disturbance. Such a system would clearly be less than ideal. One must remember, however, that modal control forces are only abstract forces, and one must take certain combinations of the modal control forces to obtain the actual actuator forces. Hence, it is conceivable that a synthesis of the modal control forces can result in actuator forces concentrated around the disturbance. For this to happen, the modal forces must reinforce each other near the disturbance and cancel each other in areas far from the disturbance. We investigate which of the two possibilities occurs in modal control of traveling waves in two types of elastic members: a string in transverse vibration and a beam in bending.

Another promising approach to wave control is direct feedback control,[3,4] in which the sensors and actuators are colocated and a given actuator force depends on the state at the same location. Such an approach is simpler to implement, as there is no need to estimate modal states to generate modal feedback control forces. In the case of traveling waves, it is clear that this approach does not call for actuators to be active in undisturbed locations in the structure. Control gains can be obtained by an approach based on a reduced-order model,[4] as described in this paper. One advantage of direct feedback control is that it is inherently stable, only drawing energy out of the structure when velocity feedback alone is used.[3] In this paper, we investigate the performance of direct feedback control in suppressing wave motion in the structures previously described and compare the results with those obtained by means of modal control.

Independent Modal-Space Control

Consider a distributed-parameter system whose behavior is governed by the partial differential equation of motion[6]

$$Lu(P,t) + m(P)\,\frac{\partial^2 u(P,t)}{\partial t^2} = f(P,t) \tag{1}$$

subject to the boundary conditions $B_i u(P,t) = 0$ $(i = 1,2,\ldots,p)$. Here, L is a linear, self-adjoint differential operator of order $2p$, where p is an integer, $u(P,t)$ is the displacement, a function of the position P and time t, $m(P)$ is the distributed mass, and $f(P,t)$ is a distributed control force. The B_i's are also linear-differential operators. The solution of the associated eigenvalue problem consists of a denumerably infinite set of eigenvalues Λ_k and corresponding eigenfunctions $\phi_k(P)$ $(k = 1,2,\ldots)$. The eigenvalues are the squares of the natural frequencies ω_k of the system, $\Lambda_k = \omega_k^2$, and the eigenfunctions

---

Presented as Paper 87-0942 at the AIAA Dynamics Specialists Conference, Monterey, CA, April 9–10, 1987; received July 9, 1987; revision received Feb. 4, 1988.

*Graduate Research Assistant, Department of Engineering Science and Mechanics; currently, Assistant Professor, Department of Aerospace Engineering and Engineering Mechanics, University of Texas, Austin, TX. Member AIAA.

†University Distinguished Professor, Department of Engineering Science and Mechanics. Fellow AIAA.

are orthogonal and can be normalized so as to satisfy

$$\int_D m(P)\phi_k(P)\phi_\ell(P)\,\mathrm{d}D = \delta_{k\ell},$$

$$\int_D \phi_k(P)L\phi_\ell(P)\,\mathrm{d}D = \Lambda_k\delta_{k\ell} = \omega_k^2\delta_{k\ell}$$

where $\delta_{k\ell}$ is the Kronecker delta.

By the expansion theorem,[6] the displacement of the structure can be expressed as

$$u(P,t) = \sum_{k=1}^{\infty}\phi_k(P)u_k(t) \tag{2}$$

where $u_k(t)$ are the modal displacements. Using the standard approach, we obtain the open-loop modal ordinary differential equations of motion

$$\ddot{u}_k(t) + \omega_k^2 u_k(t) = f_k(t), \qquad k = 1,2,\ldots \tag{3}$$

where

$$f_k(t) = \int_D \phi_k(P)f(P,t)\,\mathrm{d}D, \qquad k = 1,2,\ldots \tag{4}$$

are modal forces. In the case of the independent modal-space control method (IMSC), each modal control force depends only on the corresponding modal displacement and velocity.[2] Hence, for linear feedback,

$$f_k(t) = f_k[u_k(t),\dot{u}_k(t)] = -g_k u_k(t) - h_k\dot{u}_k(t), \quad \mathrm{k} = 1,2,\ldots \tag{5}$$

where g_k and h_k are modal control gains. Inserting Eq. (5) into Eq. (3), we obtain the independent closed-loop modal equations

$$\ddot{u}_k(t) + h_k\dot{u}_k(t) + (\omega_k^2 + g_k)u_k(t) = 0, \quad k = 1,2,\ldots \tag{6}$$

Implementation of modal control without spillover requires a distributed control having the expression

$$f(P,t) = \sum_{k=1}^{\infty} m(P)\phi_k(P)f_k(t) \tag{7}$$

If the control is to be carried out by means of m discrete point actuators, the distributed control force can be written as

$$f(P,t) = \sum_{j=1}^{m} F_j(t)\,\delta(P - P_j) \tag{8}$$

where $\delta(P - P_j)$ is a spatial Dirac delta function. Then, from Eq. (4), each modal control force is given by

$$\begin{aligned} f_k(t) &= \int_D \phi_k(P)\sum_{j=1}^{m} F_j(t)\,\delta(P - P_j)\,\mathrm{d}D \\ &= \sum_{j=1}^{m} F_j(t)\,\phi_k(P_j), \quad k = 1,2,\ldots \end{aligned} \tag{9}$$

Letting f be the vector of modal control forces and F the vector of actual control forces, we can write

$$f = BF \tag{10}$$

where the matrix $B = [B_{kj}] = [\phi_k(P_j)]$ is known as the modal participation matrix. We can obtain the vector of actual control forces from the vector of modal control forces by writing

$$F = B^{\dagger}f \tag{11}$$

where $B^{\dagger}$ is the pseudoinverse of B. If there are as many discrete actuators as controlled modes, then B is a square matrix so that, if we assume that B is nonsingular, Eq. (11) reduces to

$$F = B^{-1}f \tag{12}$$

To generate the modal control forces, we need the modal displacements and velocities. We can extract them from the displacement and velocity profiles using the expansion theorem

$$u_k(t) = \int_D m(P)\,\phi_k(P)u(P,t)\,\mathrm{d}D$$

$$\dot{u}_k(t) = \int_D m(P)\,\phi_k(P)\dot{u}(P,t)\,\mathrm{d}D \tag{13}$$

If we use n discrete sensors, then we can interpolate between the sensor measurements to obtain the approximate profiles $\hat{u}(P,t)$ and $\hat{\dot{u}}_k(P,t)$. Then we compute the estimates $\hat{u}_k(t)$ and $\hat{\dot{u}}_k(t)$ by inserting $\hat{u}(P,t)$ and $\hat{\dot{u}}_k(P,t)$ in Eq. (13). Alternatively, we note that at the sensor locations P_i,

$$u(P_i,t) = \sum_{k=1}^{\infty}\phi_k(P_i)u_k(t) = \boldsymbol{\phi}^T(P_i)\boldsymbol{u}(t), \quad i = 1,2,\ldots,n \tag{14}$$

where $\boldsymbol{\phi}^T(P_i)$ is the infinite-dimensional vector of eigenfunctions evaluated at $P = P_i$ and $\boldsymbol{u}(t)$ is the infinite-dimensional modal displacement vector. Introducing the measurement vector $\boldsymbol{y}(t)$ with components $y_i(t) = u(P,t)$, we can rewrite Eq. (14) as

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}^T\boldsymbol{u}(t) \tag{15}$$

where $\Phi = [\phi_{ki}] = [\phi_k(P_i)]$ is an $\infty \times n$ matrix. Then, if we truncate the modal displacement vector $\boldsymbol{u}(t)$ so that its dimension is equal to the number of sensors, we can estimate it from

$$\hat{\boldsymbol{u}}(t) = (B_s^T)^{-1}\boldsymbol{y}(t) \tag{16}$$

where B_s is a square truncated matrix Φ and it represents the sensor participation matrix. This is equivalent to using the lowest n modes to represent the displacement profile. Similarly, the estimated modal velocity vector is

$$\hat{\dot{\boldsymbol{u}}}(t) = (B_s^T)^{-1}\dot{\boldsymbol{y}}(t) \tag{17}$$

Finally, the modal equations of motion become

$$\ddot{u}_k + \omega_k^2 u_k = f_k = -g_k\hat{u}_k - h_k\hat{\dot{u}}_k \tag{18}$$

In this study, we use gains that minimize the performance functional

$$\begin{aligned} J = \int_0^{\infty}\Bigg\{\int_D &[m(P)\dot{u}^2(P,t) + u(P,t)Lu(P,t) \\ &+ R(P)f^2(P,t)]\,\mathrm{d}D\Bigg\}\,\mathrm{d}t \end{aligned} \tag{19}$$

where $R(P)$ is a control effort weighting function. If $R(P) = r/m(P)$, then the minimization of J can be carried out by using Eqs. (2) and (7) to decompose J into modal performance indices J_k. The latter can be minimized independently, resulting in the gains[2]

$$\begin{aligned} g_k &= -\omega_k^2 + \omega_k\,[\omega_k^2 + (1/r)]^{1/2} \\ h_k &= \{-2\omega_k^2 + (1/r) + 2\omega_k[\omega_k^2 + (1/\mathrm{r})]^{1/2}\}^{1/2}, \quad k = 1,2,\ldots \end{aligned} \tag{20}$$

In the case of a damped system, the partial-differential equation of motion is

$$Lu(P,t) + C\frac{\partial u(P,t)}{\partial t} + m(P)\frac{\partial^2 u(P,t)}{\partial t^2} = f(P,t) \tag{21}$$

where C is a differential operator. If C can be expressed as the linear combination

$$C = \alpha_1 L + \alpha_2 m(P) \tag{22}$$

where α_1 and α_2 are constants, then the open-loop modal equations of motion become

$$\ddot{u}_k + (\alpha_1\omega_k^2 + \alpha_2)\dot{u}_k + \omega_k^2 u_k = f_k, \quad k = 1,2,... \tag{23}$$

so that the open-loop equations remain uncoupled. This special case of damping is known as proportional damping. Then, if we use Eq. (5), the closed-loop modal equations are simply

$$\ddot{u}_k + (h_k + \alpha_1\omega_k^2 + \alpha_2)\dot{u}_k + (g_k + \omega_k^2)u_k = 0, \quad k = 1,2,... \tag{24}$$

Direct Feedback Control

In direct feedback control, the sensor output at a given point is multiplied directly by a control gain to yield the control force at that point. In essence, direct feedback control can be regarded as the control of a selected number of points on the distributed structure, where the points in question coincide simply with the locations of the sensors and actuators. Because this is not a modal control method, the control design need not be based on the modal equations of motion. In view of this, we consider another way of producing a truncated model. To this end, we denote the displacement outputs at the sensor locations $P = P_j$ by

$$y_j(t) = u(P_j,t), \quad j = 1,2,...,m \tag{25}$$

Then, the actuator force $F_j(t)$ at $P = P_j$ can be written as

$$F_j(t) = -g_j y_j(t) - h_j \dot{y}_j(t), \quad j = 1,2,...,m \tag{26}$$

where g_j and h_j are control gains associated with displacements and velocities, respectively. Equations (26) have the matrix form

$$F(t) = -Gy(t) - H\dot{y}(t) \tag{27}$$

where

$$G = \text{diag}[g_j], \quad H = \text{diag}[h_j] \tag{28}$$

are diagonal control gain matrices and $y(t)$ and $\dot{y}(t)$ are m-dimensional displacement and velocity output vectors, respectively. One of the issues in direct feedback control has been how to determine the gain matrices G and H in a rational manner. Clearly, the fact that G and H are diagonal places some constraints on the problem.

To produce the previously mentioned truncated model, let us assume that the motion of the structure can be described in terms of the m lowest modes, on the premise that these are the modes most likely to participate in the motion. Hence, truncating series (2), we can write

$$u(P,t) = \sum_{k=1}^{m} \phi_k(P)u_k(t) \tag{29}$$

Then, with Eqs. (3) and (9), the truncated modal equations can be written in the vector form

$$\ddot{u}(t) + \Lambda u(t) = f(t) = \Phi_m F(t) \tag{30}$$

where Λ is a diagonal matrix of eigenvalues, $u(t)$ is the m-dimensional truncated modal vector and $\Phi_m = [\phi_{kj}] = [\phi_k(P_j)]$ is the $m \times m$ truncated modal participation matrix. Moreover, inserting Eq. (29) into Eq. (25), we can write the displacement output vector in the form

$$y(t) = \Phi_m^T u(t) \tag{31}$$

so that

$$u(t) = (\Phi_m^T)^{-1} y(t) = (\Phi_m^{-1})^T y(t) \tag{32}$$

Hence, introducing Eq. (32) into Eq. (30) and multiplying on the left by Φ_m^{-1}, we obtain the desired truncated model

$$M\ddot{y}(t) + Ky(t) = F(t) \tag{33}$$

where

$$M = \Phi_m^{-1}(\Phi_m^{-1}])^T, \qquad K = \Phi_m^{-1}\Lambda(\Phi_m^{-1})^T \tag{34}$$

can be regarded as mass and stiffness matrices.

Next, we propose to use the truncated model given by Eq. (33) to determine suitable control gains. Ideally, we would like to produce optimal gain matrices G and H, although the fact that these matrices are not fully populated may cause difficulties. To pursue an optimal control policy, we first cast the equations of motion in state form. To this end, we introduce the $2m$-dimensional state vector.

$$x(t) = [y^T(t) \,|\, \dot{y}^T(t)]^T \tag{35}$$

Then, adjoining the identity $\dot{y}(t) = \dot{y}(t)$ to Eq. (33), we obtain the state equations in the vector form

$$\dot{x}(t) = Ax(t) + BF(t) \tag{36}$$

in which

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & 0 \end{array}\right] \tag{37a}$$

$$B = \left[\begin{array}{c} 0 \\ \hline M^{-1} \end{array}\right] \tag{37b}$$

are the coefficient matrices. We consider the quadratic performance index

$$J = \int_D^{\infty} (x^T Q x + F^T R F) \, dt \tag{38}$$

where

$$Q = \left[\begin{array}{c|c} K & 0 \\ \hline 0 & M \end{array}\right] \tag{39}$$

and R is a control effort weighting matrix. Then, if we consider constant gains, the optimal control is given by

$$F(t) = -R^{-1}B^T P x(t) = -Gy(t) - H\dot{y}(t) \tag{40}$$

where P satisfies the matrix algebraic Riccati equation[7]

$$Q - PBR^{-1}B^T P + A^T P + PA = 0 \tag{41}$$

As we just noted, in general G and H in Eq. (40) are fully populated. However, in certain cases, the diagonal entries in the velocity gain matrix H are several orders of magnitude larger than the off-diagonal entries. Also, the displacement gains G are often sufficiently small that their contribution to the control forces can be neglected. Both of these are true in the examples presented in this paper. Hence, we obtain something very close to direct feedback control as the solution to the optimal control problem, and the simplification in implementation resulting from using direct feedback control justifies ignoring the off-diagonal entries in the gain matrices.

An important but often neglected step in the design of any structural control system is that of simulating the performance of the controls on the actual distributed structure rather than on the model used for designing the controls. We do this in the next section to check the simplifications made, which include using an undamped reduced model to obtain control gains and neglecting the less significant entries in the gain matrices to implement direct feedback control.

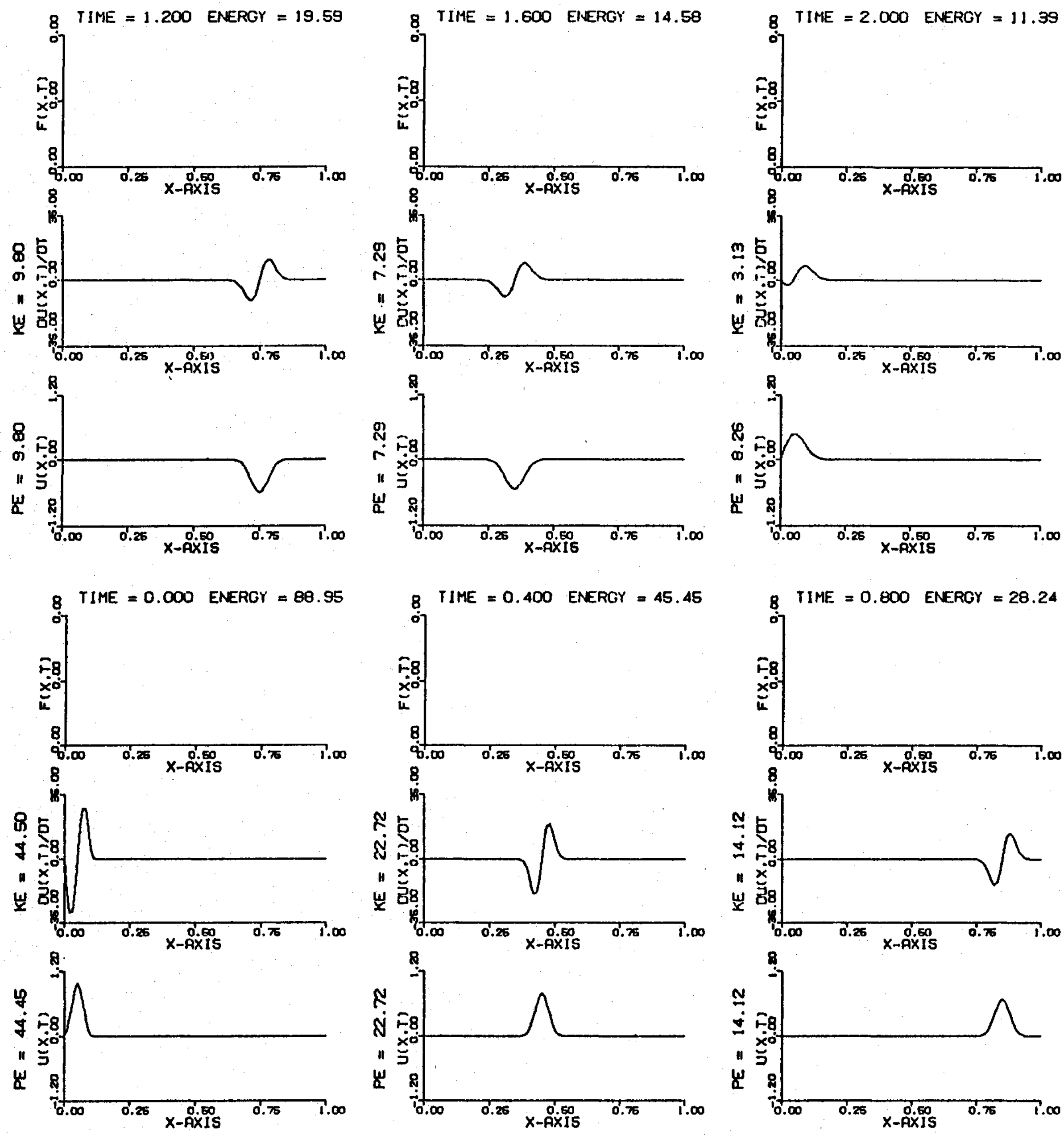

Fig. 1 Uncontrolled damped wave motion in a string.

Control of Traveling Waves

In this section, we use IMSC and direct feedback to control traveling waves in flexible structures. In each case, we begin with initial conditions describing a single, localized traveling disturbance in the structure. Because the modes of a distributed structure form a complete set in energy, any disturbance can be expressed as a linear combination of these modes, provided a sufficiently large number is included. Hence, we use a modal approach to model the structures studied here, with as many as 80 modes included in each simulation, to make sure that each system is modeled adequately. Note that a relatively large number of modes is required to model traveling-wave behavior in a flexible structure when the length of the traveling wave is small compared to the length of the structure.

We consider first the wave motion in a second-order system, such as a string in transverse vibration, a bar in axial vibration, or a shaft in torsional vibration. Then, we consider traveling waves in fourth-order systems, such as a beam in bending vibration. In the first case, if the system is undamped, the waves travel through the system without changing shape. In the second, the system is dispersive, so that the wave changes shape as it travels along the beam. For both types of systems, we assume that there is inherent internal damping, and that this damping is proportional to the local rate of strain in the material. In each case, we consider first the globally optimal solution to the control problem obtained by using distributed actuators. Although implementation of control by means of distributed actuators is not within the state of the art, the globally optimal solution is valuable because it gives qualitative information about what the optimal control force distribution should be. Then, we consider control of the wave motion by means of a finite number of discrete actuators using the modal and direct feedback approaches. In the first case, only a limited number of the lower modes will be controlled. A comparison of the results obtained will demonstrate the effectiveness of the two methods in controlling waves with only a finite number of discrete actuators.

Second-Order Systems

We consider a second-order system in the form of a string in transverse vibration. If we assume that the system is undamped, the free vibration is governed by the partial-differen-

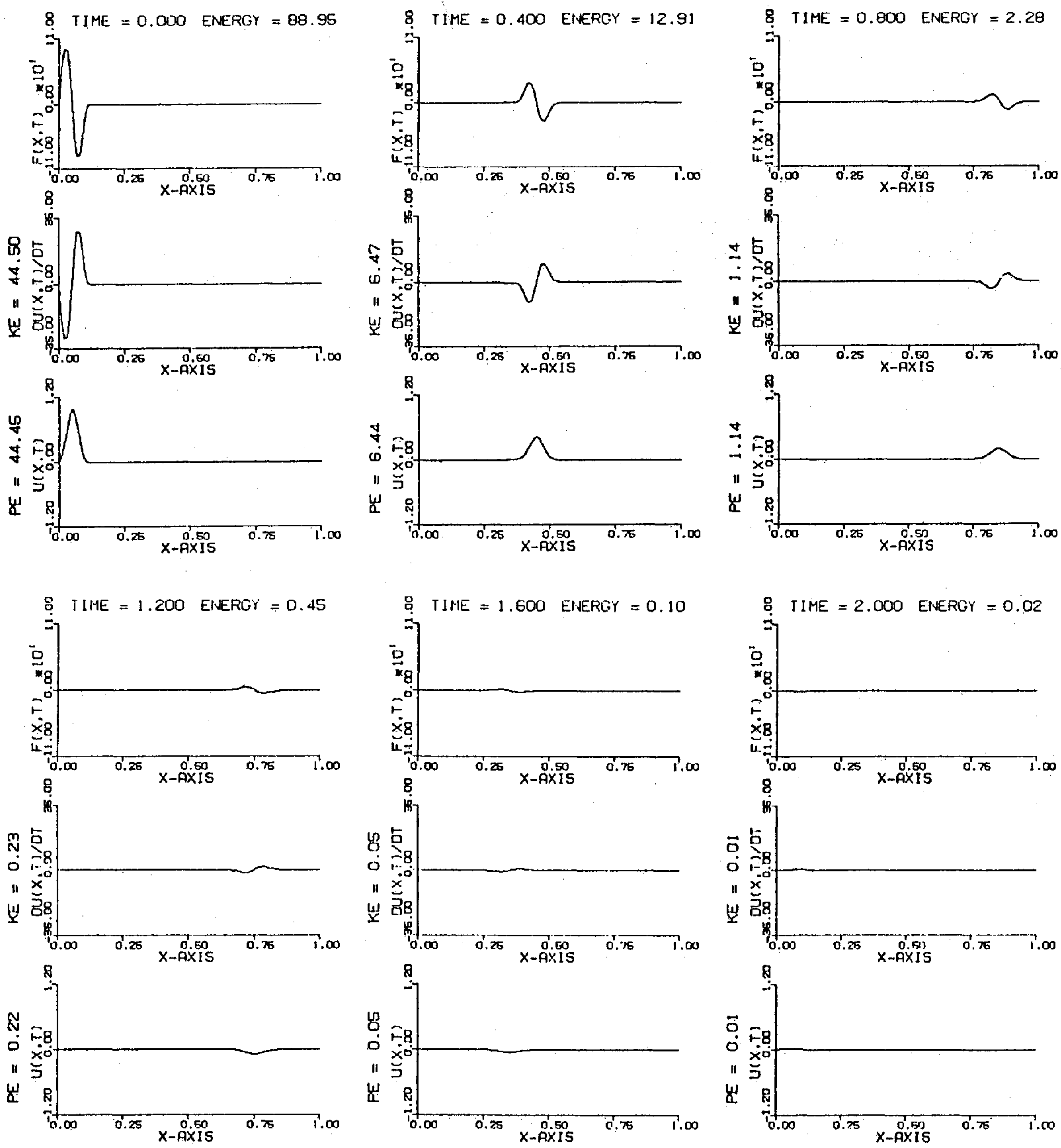

Fig. 2 Globally optimal (distributed) control of wave motion in a string.

tial equation[8]

$$-T\frac{\partial^2 u(x,t)}{\partial x^2}+m\frac{\partial^2 u(x,t)}{\partial t^2}=0 \tag{42}$$

where $u(x,t)$ is the transverse displacement, T the tension, and m the mass per unit length. Here, the differential operator L is equal to $-T\partial^2/\partial x^2$. It is assumed that both T and m are constant.

If the string is of infinite length, it is easy to show that the solution of Eq. (42) can be written in the form[8]

$$u(x,t)=F_1(x-\nu t)+F_2(x+\nu t) \tag{43}$$

where F_1 and F_2 are wave profiles traveling to the right and left, respectively, with the wave velocity

$$\nu=\sqrt{T/m} \tag{44}$$

If the string is finite and fixed at both ends, then the solution $u(x,t)$ of Eq. (42) must satisfy the boundary conditions

$$u(0,t)=u(L,t)=0 \tag{45}$$

It can be shown[8] that the natural frequencies are

$$\omega_k=k\pi\sqrt{T/mL^2},\qquad k=1,2,\ldots \tag{46}$$

and the associated normalized eigenfunctions are

$$\phi_k(x)=\sqrt{2/mL}\,\sin(k\pi x/L),\qquad k=1,2,\ldots \tag{47}$$

According to the expansion theorem, Eq. (2), the displacement of the string can be represented by a linear combination of these eigenfunctions. Alternatively, at any instant in time, the motion can be described by Eq. (43) in terms of traveling waves, as long as the boundary conditions are satisfied. These boundary conditions determine how the wave is reflected at the boundaries of the string.

Next, we add an external distributed force and distributed damping that is proportional to the local strain rate, so that the partial-differential equation of motion becomes

$$-T\frac{\partial^2 u(x,t)}{\partial x^2}-C\frac{\partial^3 u(x,t)}{\partial x^2\partial t}+m\frac{\partial^2 u(x,t)}{\partial t^2}=f(x,t) \tag{48}$$

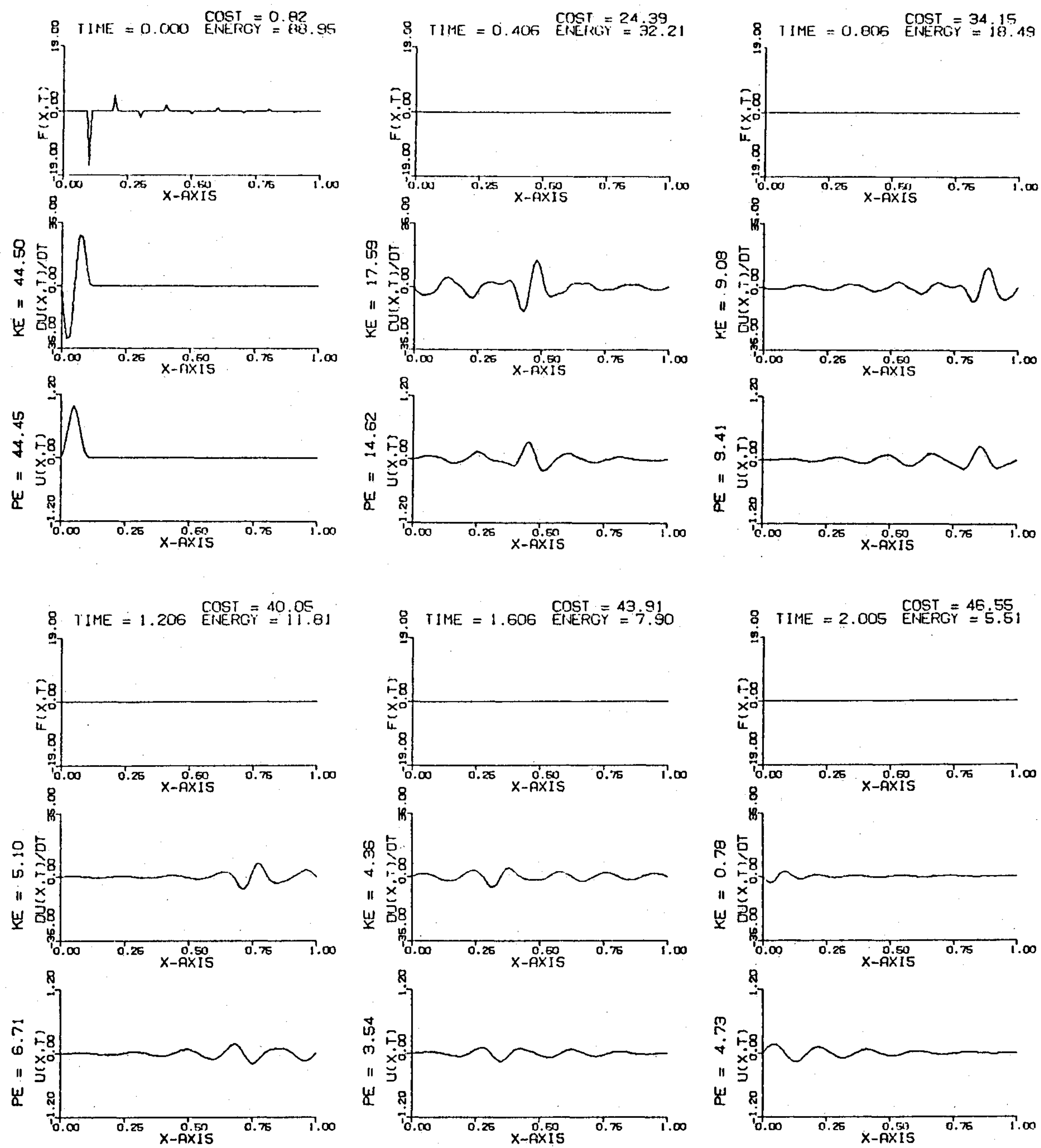

Fig. 3 Modal control of a wave in a string using discrete actuators.

where C is assumed to be constant. Because damping is of the proportional type,[6] the eigenfunctions of the damped system are the same as the eigenfunctions of the undamped system, although the eigenvalues are different. Hence, inserting Eq. (2) with $P = x$ into Eq. (48), multiplying by $\phi_\ell(x)$, integrating over the length of the string, and making use of the orthogonality relations, we obtain the independent ordinary differential equations of motion.

$$\ddot{u}_k(t) + (C\omega_k^2/T)\dot{u}_k(t) + \omega_k^2\, u_k(t) = f_k(t),$$

$$k = 1,2,... \tag{49}$$

where

$$f_k(t) = \int_0^L \phi_k(x) f(x,t)\,\mathrm{d}x$$

is the kth modal force. Here, we note that the damping factor is proportional to the natural frequency, $\zeta_k = (C/2T)\,\omega_k$ ($r = 1,2,...$). Hence, we expect the higher modes to decay more rapidly than the lower modes, which is confirmed by the observed behavior.

One way of generating a traveling wave in a string is by moving one of the ends. If this is done to the left end, the boundary conditions become

$$u(0,t) = w_1(t) \tag{50a}$$

$$u(L,t) = 0 \tag{50b}$$

where $w_1(t)$ is the vertical displacement of the left end. To represent the motion of the string in terms of the modes in Eq. (47), the total motion can be regarded as consisting of a pseudostatic response to the time-varying end condition, Eq. (50a), and modal motion relative to this pseudostatic response. This is a standard procedure,[8] and the development is straightforward but lengthy. Because we are merely using this approach to generate initial conditions, we dispense with the details and simply present the modal initial conditions for

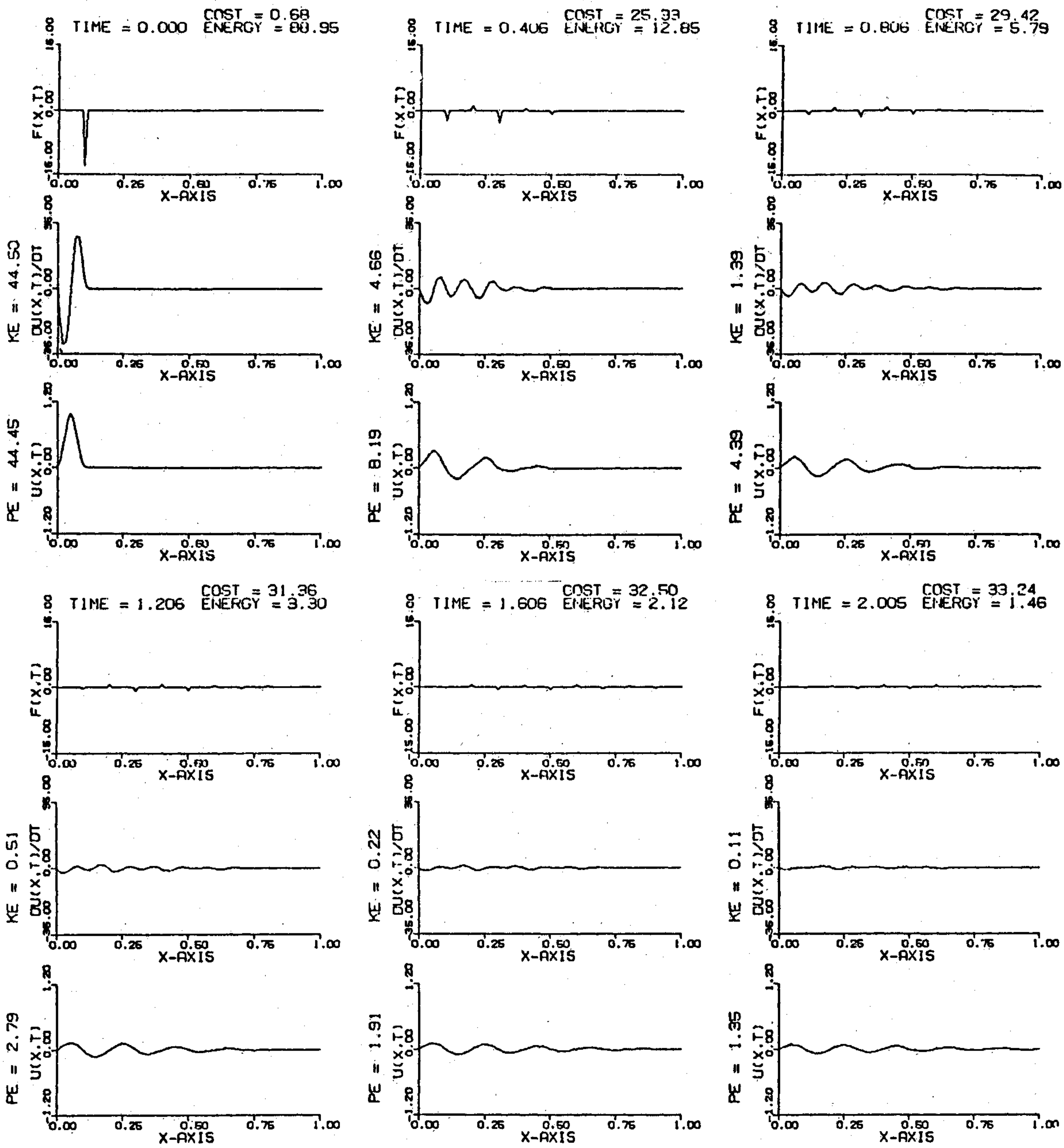

Fig. 4 Modal contributions to the actuator forces for the string at $t = 0$ and the resulting sum.

the string with a support motion of the form

$$w_1(t) = \begin{cases} (1/2)[1-\cos(2\pi t/t_0)], & -t_0 \le t \le 0 \\ 0, & t < -t_0, \quad t < 0 \end{cases} \tag{51}$$

In this case, the initial conditions become

$$u_k(0) = \frac{2\sqrt{2\pi}}{k[\omega_k^4 t_0^4 - 2(2\pi)^2\omega_k^2 t_0^2(1-2\zeta_k^2) + (2\pi)^4]} \times [(1-e^{-\zeta_k\omega_k t_0}\cos\omega_{dk}t_0)(4\pi^2-\omega_k^2 t_0^2) + (\zeta_k/\sqrt{1-\zeta_k^2})(4\pi^2+\omega_k^2 t_0^2)e^{-\zeta_k\omega_k t_0}\sin\omega_{dk}t_0] \tag{52a}$$

$$\dot u_k(0) = -\frac{2\sqrt{2\pi}}{k[\omega_k^4 t_0^4 - 2(2\pi)^2\omega_k^2 t_0^2(1-2\zeta_k^2) + (2\pi)^4]} \times \{8\pi^2\zeta_k\omega_k(1-e^{-\zeta_k\omega_k t_0}\cos\omega_{dk}t_0) + (\omega_k/\sqrt{1-\zeta_k^2})[4\pi^2(2\zeta_k^2-1)+\omega_k^2 t_0^2] \times e^{-\zeta_k\omega_k t_0}\sin\omega_{dk}t_0\} \tag{52b}$$

where

$$\omega_{dk} = \omega_k\sqrt{1-\zeta_k^2} \tag{53}$$

For simplicity, we choose the string to have unit length, tension, and mass density. From Eq. (44), this gives a wave velocity equal to unity. Hence, to generate a ripple of width λ, we choose t_0 in Eq. (51) equal to λ.

We base our evaluation of control performance partially on the performance index given by Eqs. (19) and (38). Noting that Eq. (19) cannot be evaluated when the control force distribution consists of Dirac delta functions, we use a control force distribution of the form

$$f(x,t) = \sum_{i=1}^{m} F_i(t)\,\gamma(x-x_i) \tag{54}$$

where

$$\gamma(x-x_i) = \frac{1}{\epsilon}\left\{U\left[x-\left(x_i-\frac{\epsilon}{2}\right)\right] - U\left[x-\left(x_i+\frac{\epsilon}{2}\right)\right]\right\} \tag{55}$$

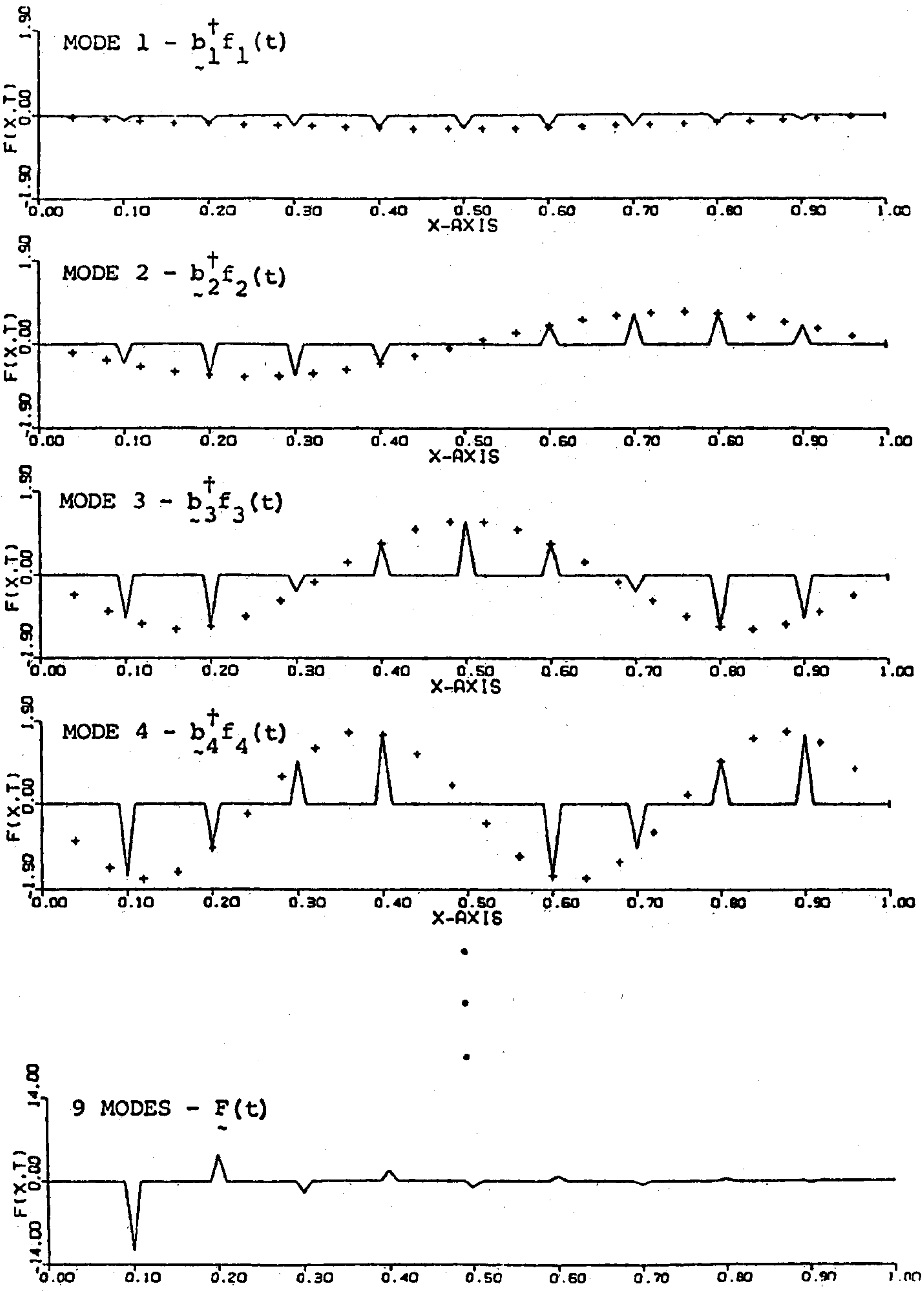

Fig. 5 Direct feedback control of a wave in a string.

with $U(x)$ as the Heaviside step function, so that $\gamma(x - x_i)$ is simply a unit rectangular force of width ϵ, centered at x_i. In these examples, we choose $\epsilon/L = 0.01$. Because

$$\int_0^L rf^2(x,t)\,\mathrm{d}t = \frac{r}{\epsilon}\sum_{i=1}^{m} F_i^2(t) \tag{56}$$

we can evaluate the cost of the control effort associated with these discrete actuators by forming the product F^TRF in Eq. (38), where the matrix R is simply given by $R = (r/\epsilon)I$, in which I is the identity matrix. Finally, we must replace B in Eq. (12) and Φ_m in Eq. (30) by a modified modal participation matrix $\hat{\Phi}$ with the elements

$$\hat{\Phi}_{kj} = \int_0^L \phi_k(x)\gamma(x - x_j)\,\mathrm{d}x = \int_{x_j-\epsilon/2}^{x_j+\epsilon/2} \sqrt{\frac{2}{mL}}\sin\frac{k\pi x}{L}\left(\frac{1}{\epsilon}\right)\mathrm{d}x$$
$$= \frac{2L}{k\pi\epsilon}\,\phi_k(x_j)\,\sin\frac{k\pi\epsilon}{2L} \tag{57}$$

Figure 1 shows the motion of the string with the foregoing initial conditions and $\lambda = 0.1L$. Here, 80 modes were used to model adequately the actual distributed string. The value of C was chosen so as to give 0.1% damping in the fundamental mode, and no control forces were applied. The effect of damping is to decrease the energy in the highest modes very quickly, so that the disturbance profile loses its initial sharpness and its amplitude decreases as the wave travels along the string.

Next, we consider a distributed control force with a control effort weighting factor of $r = 0.2$ in the performance index, Eq. (19). All the modeled modes are controlled. The results are shown in Fig. 2. We observe from Fig. 2 that the control is localized at the wave, although the control force is a linear combination of modal forces and each of the modal forces is distributed over the entire domain. This demonstrates that IMSC in its purest form, with distributed actuators, can control localized disturbances quite satisfactorily, because the control force tends to concentrate around the disturbance and

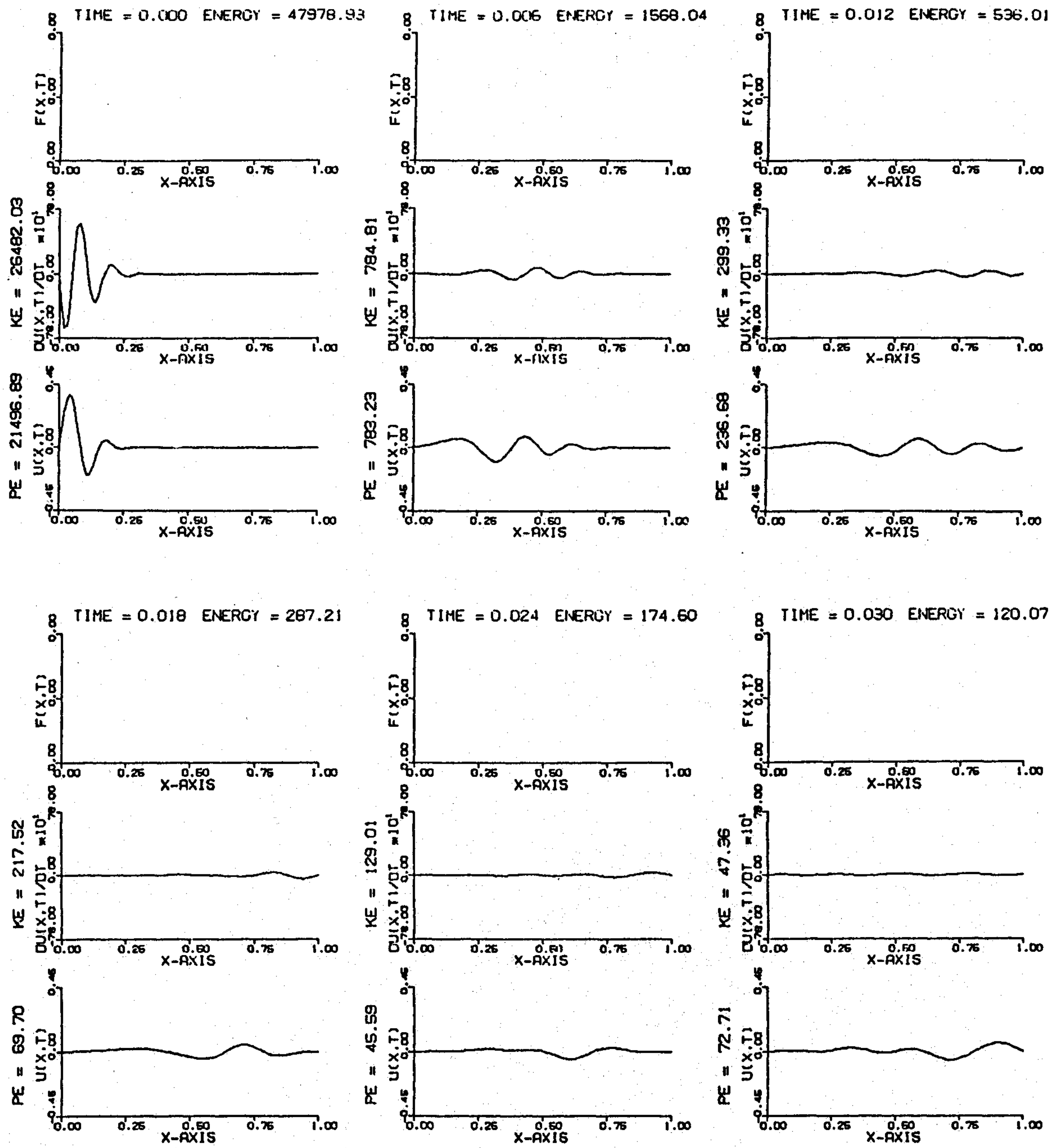

Fig. 6 Uncontrolled damped wave motion in a beam.

it travels with the wave. The same conclusion was reached in Ref. 1. We also observe from Fig. 2 that the optimal control force is very nearly equal to a scalar multiple of the velocity, which is consistent with the control objective, namely energy dissipation. This control force causes the wave to essentially retain its shape as the amplitude decreases. We can vary the rate of decay of the wave by varying r. In this example, we selected the value of r so as to enable us to monitor the effect of the controls on the system as the wave travels one complete roundtrip on the string. In general, r represents a penalty on the control effort and is chosen by the analyst so as to produce desired system performance.

Figure 3 shows results obtained by using nine discrete actuators and 19 discrete sensors, all equally spaced, to control the lowest nine modes of the string. Here, $r = 0.002$. The sensors measure the actual displacement and velocity of the string at each sensor location. Then, these measurements are used in conjunction with Eqs. (16) and (17) to estimate the corresponding modal displacements and velocities. The use of more sensors than actuators allows much of the motion due to uncontrolled modes to be filtered out. The modal control forces are calculated from the estimated modal displacements and velocities using the gains prescribed by Eqs. (20). Then, the actual actuator forces are calculated using Eq. (12). In this example, we continue to model the lowest 80 modes, so that we expect to see residual energy in the uncontrolled modes, observation spillover from the uncontrolled modes, and control spillover into the uncontrolled modes, at least to some degree. Here, we still consider the effects of damping as in the previous two cases. The use of discrete actuators causes the wave to lose its initial smooth shape with time, although the disturbance is still identifiable as it travels. If we examine the plot corresponding to $t = 0$, it is clear that, when the disturbance in the system is localized, the control force tends to be concentrated in the neighborhood of the disturbance. Comparing the rate of energy dissipation with the damped but uncontrolled case of Fig.1, we observe that controlling only the lowest nine modes increases the energy dissipation substantially. In the time increment between $t = 0$ and $t = 0.4$, damping causes a 49% loss of energy in the uncontrolled case,

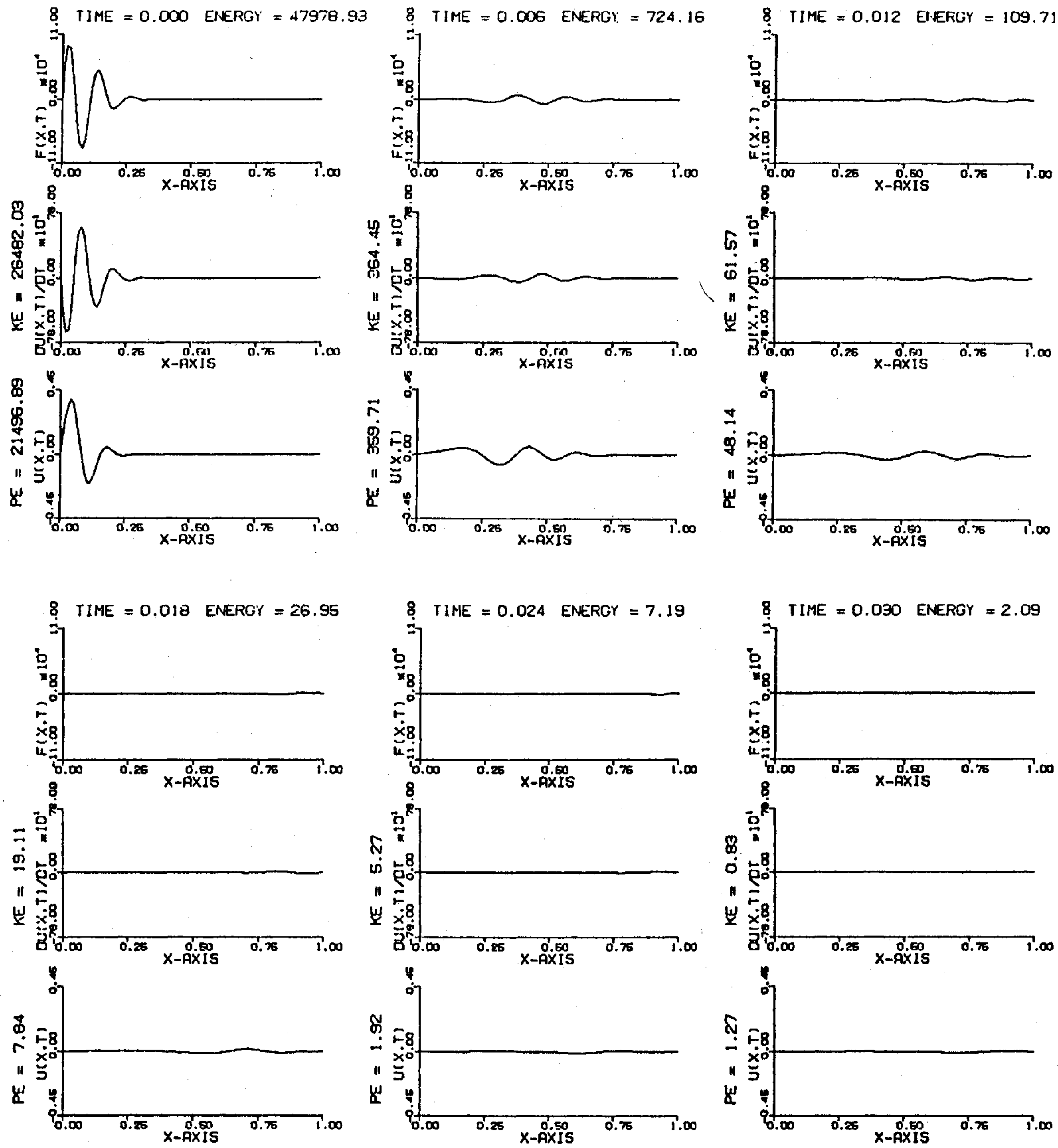

Fig. 7 Globally optimal (distributed) control of wave motion in a beam.

whereas the discrete-actuator controls dissipate an additional 14% by operating on the lowest nine modes. As time progresses, the controls become essentially inactive, indicating that motion in the lowest nine modes has been annihilated. The strain-rate damping then causes the remaining energy to decay. At $t = 4.35$, the energy in the string decreases to a level of 1% of the initial energy, with a performance index of 52.33.

Figure 4 shows plots of the modal contributions $b_k^\dagger f_k(t)$ to the actuator force vector $F(t)$ at $t = 0$ where the vectors $b_k^\dagger$ are the columns of $B^\dagger$ in Eq. (11), or of B^{-1} in Eq. (12). In actuality, the actuator force vector was computed by using the matrix $\hat{\Phi}$ with components given by Eqs. (57). Also, in each of these plots, we have sketched the corresponding mode shape to give an idea of the mode contributions if the actuators were distributed devices instead of point actuators. The last plot represents $\Sigma_{k=1}^{m} b_k^\dagger f_k(t)$ at $t = 0$, which is recognized from Eq. (11) as the actual actuator force vector $F(t)$ at $t = 0$. This figure brings out the fact that these modal forces tend to cancel each other out at points far away from the disturbance, although individually these modal forces have nonzero amplitude at such points. Hence, the actual forces, as exerted by the actuators, tend to be concentrated in the vicinity of the disturbance.

Figure 5 contains plots of the controlled response of the string when direct feedback is used. Again, $r = 0.002$. Here, nine actuators and only nine colocated sensors are used, all equally spaced. As expected, the actuators are only active exactly at the location of the disturbance, and no control effort is expended where there is no disturbance In IMSC, this is only approximately true, and there is some wasted control effort due to the actuator activity away from the disturbance. Another difference is that the actuators continue to be active longer than in IMSC, because they respond to motion in any mode and not just to the few modes targeted for control. In this case, the energy in the string was reduced to only 1% of its initial value at $t = 2.62$ with a cost of only 33.96. Noting that the cost for IMSC was 57% higher, we conclude that the direct feedback approach is clearly superior. This is due to the fact that direct feedback does not limit its attention to the lower modes, and its control effort is more strictly confined to the neighborhood of the disturbance. Further experiments indicate that as the level of structural damping decreases, the

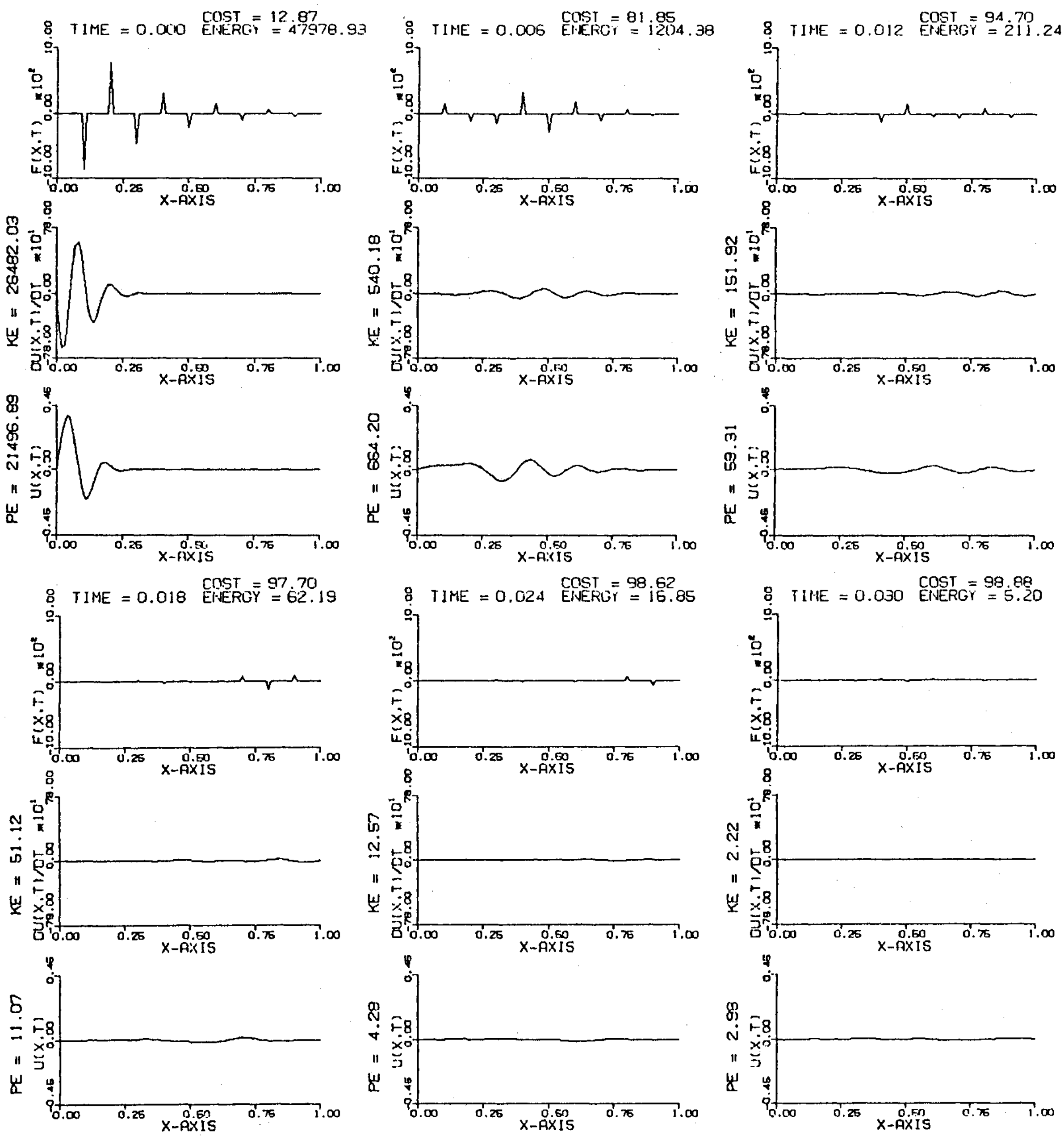

Fig. 8 Modal control of a wave in a beam using discrete actuators.

superiority of direct feedback increases. This occurs in spite of the fact that only half as many sensors are required for direct feedback control.

Fourth-Order Systems

The motion of beams in undamped free vibration is governed by the fourth-order partial-differential equation[8]

$$EI\frac{\partial^4 u(x,t)}{\partial x^4} + m\frac{\partial^2 u(x,t)}{\partial t^2} = 0, \qquad 0<x<L \tag{58}$$

where EI is the bending stiffness and m is the mass per unit length, both assumed to be constant. Here, $L = EI\partial^4/\partial x^4$. Equation (58) admits a solution in the form of the wave motion[8]

$$u(x,t) = \cos\frac{2\pi}{\lambda}(x - \nu t) \tag{59}$$

where λ is the wavelength and

$$\nu = (2\pi/\lambda)\sqrt{EI/m} \tag{60}$$

is the wave velocity. Hence, if a given wave profile is resolved into sinusoidal components by Fourier analysis, each wave component will travel with a different velocity. It follows that the wave profile changes shape as it travels, so that the beam is dispersive.[8]

If the beam has pinned ends, then the displacement must satisfy the boundary conditions

$$u(0,t) = u(L,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} = 0 \tag{61}$$

The natural frequencies are

$$\omega_k = (k\pi)^2\sqrt{EI/mL^4}, \qquad k = 1,2,... \tag{62}$$

and the associated normalized eigenfunctions are the same as for the string, Eq.(47).

In the presence of distributed damping proportional to the local strain rate and a distributed control force, the partial-differential equation of motion becomes[9]

$$EI = \frac{\partial^4 u(x,t)}{\partial x^4} + C\frac{\partial^5 u(x,t)}{\partial x^4 \partial t} + m\frac{\partial^2 u(x,t)}{\partial t^2} = f(x,t) \tag{63}$$

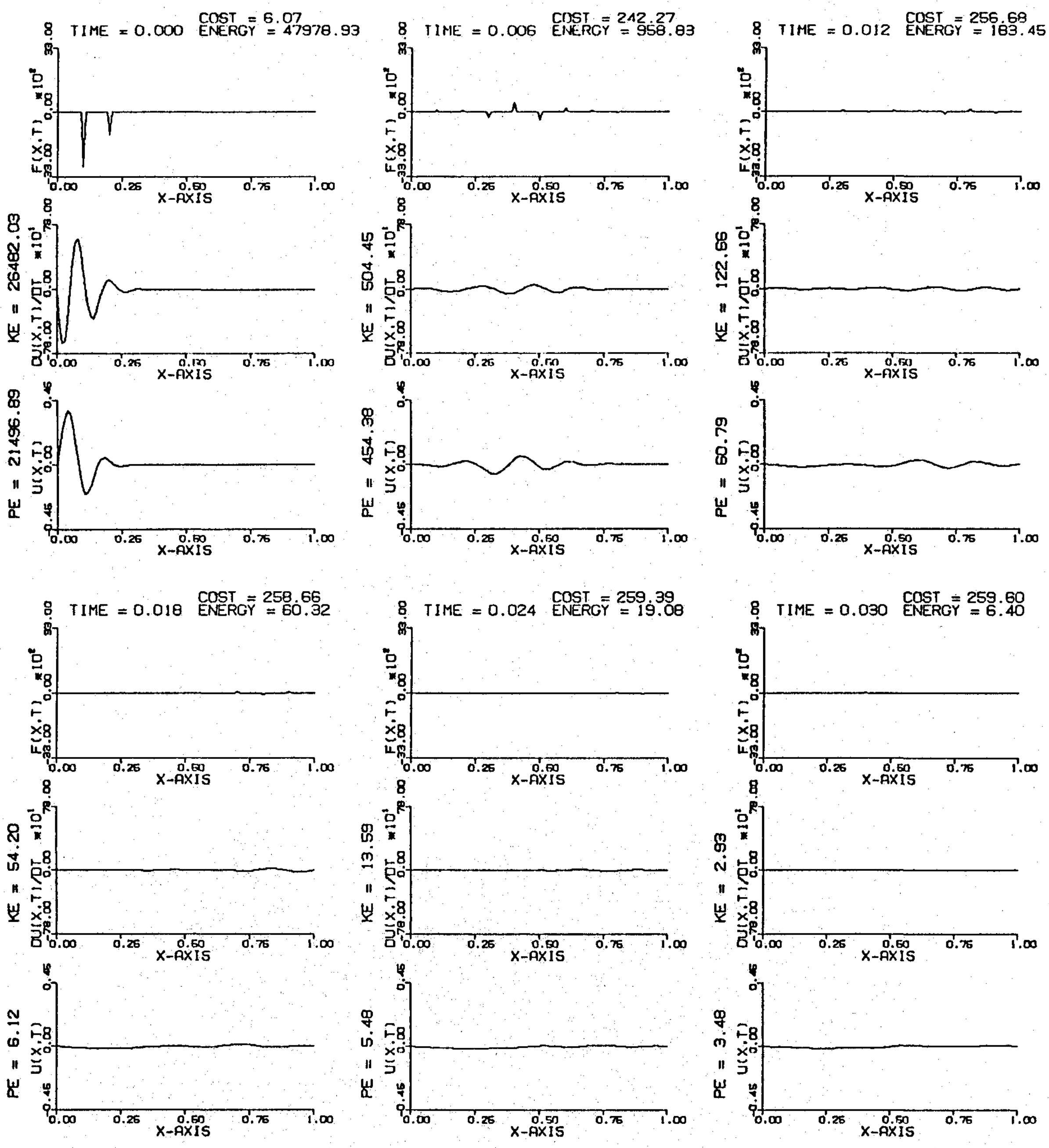

Fig. 9 Direct feedback control of a wave in a beam.

where C is the damping coefficient. The modal equations of motion can be obtained by the same approach as for the string, and the equations are nearly identical to Eqs. (49). However, in this case, the damping factors are given by $\zeta_k = (C/2EI)\omega_k$, so that the damping factor for each mode is proportional to the square of the mode number, because it is still proportional to the natural frequency. Hence, for a beam with strain-rate damping, the higher modes decay much faster than the lower modes.

We generate a traveling wave on this pinned-pinned beam by imparting to the left end the vertical motion given by Eq. (51). This results in the same initial conditions as in Eqs. (52). Again, we set the values of m, EI, and L equal to 1. However, in this case, to get a "ripple" of nominal width λ, we must choose $t_0 = \lambda^2/2\pi$ because this is how long a wave of wavelength λ takes to "pass through" the left end onto the beam, according to Eq. (60), and λ is the wavelength of the dominant Fourier component of such a ripple. Again, we choose $\lambda = 0.1L$, and we use 80 modes to model the behavior of the beam.

Figure 6 shows the uncontrolled motion of the beam with these initial conditions, with C chosen so that the damping factor in the first mode is again equal to about 0.1%. The dispersive nature of the beam is obvious from the change in the wave profile. Indeed, even at the start, the wave profile has a different appearance from that of the string, as some of the higher Fourier components break away more quickly. However, the nominal width that we chose for the ripple is still evident.

An aspect of this damped wave motion worthy of particular note is the rapid initial dissipation of energy in the higher modes. Indeed, at $t = 0.006$, only about 1/30 of the initial energy remains. At $t = 0.012$, only about 1/90 remains, and from there on the decay is more gradual, as the lower modes take much longer to decay.

Figure 7 shows the globally optimal control with distributed actuators. Again, the control force looks like the negative of the velocity. Here, $r = 0.0001$. Figure 8 contains plots of IMSC with nine discrete actuators and 19 discrete sensors. identical to those used in the case of the string, with $r = 0.0001$. Again, we see that the control forces tend to be concentrated near the disturbance, although there is some effort expended elsewhere on the beam. We note that the actuators are still working when the beam has almost come to rest, whereas in the case of the string, there was still a significant amount of energy in the system when the actuators stopped working. This indicates that the lower modes in the beam linger the longest because the higher ones are damped out so quickly. The performance index at the completion of the control task is 99.00

In Fig. 9, we see the response of the direct feedback control system, which again uses nine colocated pairs of actuators and sensors. Here again, the actuators away from the wave are completely inactive, as expected. But comparing the scale on the force axis with the one in Fig. 8, we see that the actuator forces in this case are much greater, at least initially. The performance index for the direct feedback approach is 259.67, or 162% higher than that obtained with IMSC. The explanation for this is that considerable effort was expended in response to the high velocities initially present in the beam, and a large portion of these high velocities was contributed by higher modes that would have simply decayed rapidly on their own. In the case of IMSC, the contributions of higher modes were filtered out rather effectively so that the actuators could control the lower nine modes. Hence, in the case of the beam, IMSC proved to be the superior control technique because of the greater separation of natural frequencies and damping factors in the fourth-order case. The model used to design the direct feedback control system proved to be too simplistic to account for the variety of modes with their widely varying damping factors. Indeed, because direct feedback control responds to the entire sensed motion without attempting to discriminate between modal contributions, it tends to be less satisfactory than modal control when only a few modes require control, regardless of the model used to obtain the control gains.

Conclusions

This paper considers two approaches, IMSC and direct feedback control, for controlling traveling waves in structures. It is demonstrated that when there is a small amount of material damping in a structure with widely spaced natural frequencies, IMSC is attractive because control can be concentrated on the lower modes while higher modes are damped out more quickly on their own, as the modal damping factors are proportional to modal frequencies. We recall that in IMSC the number of controlled modes is equal to the number of actuators. However, when the modal frequencies are closely spaced, as in the case of the string, direct feedback control becomes more attractive because the number of modes from which energy must be dissipated is much greater than the number of actuators. It is also demonstrated that in IMSC, as in direct feedback control, the actuator forces do tend to concentrate near a localized disturbance such as a traveling wave, in spite of the fact that IMSC represents a global approach.

Acknowledgement

Sponsored by the AFOSR Research Grant 83-0017 monitored by Dr. A. K. Amos, whose support is greatly appreciated.

References

[1]Meirovitch, L. and Bennighof, J. K., "Modal Control of Traveling Waves in Flexible Structures," *Journal of Sound and Vibration,* Vol. 111, 1986, pp. 131–144.

[2]Meirovitch, L. and Baruh, H., "Control of Self-Adjoint Distributed-Parameter Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 5, Jan.–Feb. 1982, pp. 60–66.

[3]Balas, M. J., "Direct Velocity Feedback Control of Large Space Structures," *Journal of Guidance and Control,* Vol. 2, May–June 1979, pp. 252–253.

[4]Meirovitch, L. and Bennighof, J. K., "Active Vibration Control of a Distributed System with Moving Support," American Society of Mechanical Engineers Paper 86-WA/NCA-25, Dec. 1986; also "Active Vibration Control of a Distributed System with Moving Support," *Journal of Vibration Acoustics, Stress and Reliability in Design,* Vol. 110, April 1988, pp 246–253.

[5]von Flotow, A. H., "Traveling Wave Control for Large Spacecraft Structures," *Journal of Guidance, Control, and Dynamics,* Vol. 9, July–Aug. 1986, pp. 462–468.

[6]Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijthoff & Noordhoff, The Netherlands, 1980.

[7]Kirk, D. E., *Optimal Control Theory,* Prentice-Hall, Englewood Cliffs, NJ, 1970.

[8]Meirovitch, L., *Analytical Methods in Vibrations,* Macmillan, New York, 1967.

[9]Clough, R. W. and Penzien, J., *Dynamics of Structures,* McGraw-Hill, New York, 1975.